Kusner's Conjecture in ℓ_1^5 and ℓ_1^6

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Abstract

Kusner's conjecture asserts that $e(\ell_1^n) = 2n$, where $e(X)$ denotes the maximum cardinality of an equilateral subset of X. The cases $n \leq 3$ were established by Bandelt, Chepoi, and Laurent, and the case $n = 4$ by Koolen, Laurent, and Schrijver in 2000. We prove the next two cases, $e(\ell_1^5) = 10$, $e(\ell_1^6) = 12$. The proof combines a geometric clipping reduction with reciprocal weighted energy identities and two estimates involving coordinate spans and endpoint masses, reducing the problem to the exclusion of a two parameter scalar region. Both results are formally verified in Lean.

1 Introduction

For a positive integer n, let ℓ_1^n denote $\mathbb{R}^n$ equipped with the norm $\|x\|_1 = \sum_{j=1}^n |x_j|$. A subset $S \subseteq \ell_1^n$ is *equilateral* if there is a number $\rho > 0$ such that $\|x - y\|_1 = \rho$ for all distinct $x, y \in S$. The *equilateral dimension* $e(\ell_1^n)$ is the largest cardinality of an equilateral subset of ℓ_1^n.

Let $e_1, \ldots, e_n$ be the standard basis vectors, and consider

$$\left\{\pm\tfrac{1}{2}e_j : 1 \leq j \leq n\right\}.$$

These $2n$ points have pairwise distance one. Thus $e(\ell_1^n) \geq 2n$.

Kusner's conjecture, recorded by Guy in 1983 [1], asserts that this construction is optimal in every dimension:

$$e(\ell_1^n) = 2n \qquad (n \geq 1).$$

Bandelt, Chepoi, and Laurent [2] established the conjecture for $n \leq 3$, and Koolen, Laurent, and Schrijver [3] proved the case $n = 4$. In arbitrary dimension, Alon and Pudlák [4] proved the asymptotic upper bound

$$e(\ell_1^n) = O(n \log n).$$

For $1 < p < \infty$, the analogous conjecture asserted that $e(\ell_p^n) = n + 1$ in every dimension. Swanepoel [5] showed that, for every fixed $1 < p < 2$, one has $e(\ell_p^n) > n+1$ for all sufficiently large n. In 2026, Ge, Xu, and Zhou [6] proved this equality for $2 \leq p \leq 4$. Chalmers [7] disproved it for $p = 5$ by constructing an equilateral set of 58 points in ℓ_5^{56}, and Xiong [8] obtained counterexamples for every $p > 4$. For $p = 1$, we establish Kusner's conjecture in dimensions five and six.

Theorem 1.1 (Main theorem) *Kusner's conjecture holds in dimensions five and six:*

$$e(\ell_1^5) = 10, \qquad e(\ell_1^6) = 12.$$

Our argument uses the nested cut representation of rectilinear metrics employed by Koolen, Laurent, and Schrijver [3]. Ordering the values in each coordinate expresses each coordinate distance as the sum of the intervening gaps. Each gap determines a cut separating the points on its two sides, and these cuts form a nested family for each coordinate.

To prove Kusner's conjecture in dimension n, it suffices, by scaling and passing to a subset, to exclude a unit equilateral family of $2n + 1$ points.

For such a family, terminal clipping produces a positive star metric: each point has a positive radius, and the distance between two distinct points is the sum of their radii. Taking reciprocal radii as weights gives coordinate energies whose average is exactly 2. Span and endpoint estimates give two upper bounds for this average in terms of two parameters M and w. Comparing these bounds with the exact average yields two necessary scalar inequalities.

Increasing the dimension enlarges the admissible parameter region and weakens both inequalities. It therefore suffices to show that they have no simultaneous solution in the admissible region in dimension six; dimension five then follows by monotonicity. In a later remark, we give a shorter direct proof in dimension five using the span estimate alone.

The proof of Theorem 1.1 is deferred to the end of Section 6. Both results are formally verified in Lean; the formalization is described in Section 7.

2 From equilateral sets to star metrics

Write $d_1(u, v) = \|u - v\|_1$ for the distance in ℓ_1^n. A gap adjacent to a unique coordinate minimum or maximum separates the extremal label from all others, so it contributes only to distances involving that label. Removing these *terminal gaps* reduces each pairwise distance by the sum of the two labels' total displacements.

Definition 2.1 (Terminal clipping and residual radii) Let $x_1, \dots, x_m \in \mathbb{R}^n$, with $m \geq 3$. For each coordinate j, write its original values in nondecreasing order, with multiplicity, as

$$a_{1j} \leq a_{2j} \leq \cdots \leq a_{mj}.$$

The *terminal clipping* projects each coordinate value x_{ij} onto the interval $[a_{2j}, a_{m-1,j}]$, giving points $y_1, \dots, y_m$ with

$$y_{ij} = \min\{a_{m-1,j}, \max\{a_{2j}, x_{ij}\}\}.$$

Both endpoint replacements use the original coordinate values and are made simultaneously. Define the *total displacement* and *residual radius* of label i by

$$s_i = \sum_{j=1}^{n} |x_{ij} - y_{ij}|, \qquad r_i = \frac{1}{2} - s_i.$$

Proposition 2.2 (Clipping reduction) *Let $x_1, \dots, x_{2n+1}$ be a unit equilateral family in ℓ_1^n, where $n \geq 2$. Its terminal clipping satisfies*

$$d_1(y_i, y_k) = r_i + r_k \qquad (i \neq k).$$

Moreover:

(1) *In every coordinate, the minimum and maximum of the clipped values are each attained by at least two labels.*

(2) *Every label attaining an original coordinate minimum or maximum remains at the corresponding minimum or maximum after clipping.*

(3) *There is an untouched label p, for which $y_p = x_p$ and $r_p = 1/2$.*

(4) *For all labels i, k and coordinates j,*

$$|y_{ij} - y_{kj}| \le \min\{r_i, r_k\}.$$

(5) *For every label i,*

$$\frac{1}{2(n-1)} \le r_i \le \frac{1}{2}.$$

Proof, Each removed gap reduces precisely the distances involving its unique extremal label. The gaps removed at label i have total length s_i, so for $i \neq k$,

$$d_1(y_i, y_k) = d_1(x_i, x_k) - s_i - s_k = 1 - s_i - s_k = r_i + r_k.$$

With $m = 2n+1$, the first two ordered entries in each coordinate clip to the residual minimum a_{2j}, and the last two ordered entries to the residual maximum $a_{m-1,j}$. Every original endpoint label remains at the corresponding residual end. If $a_{2j} = a_{m-1,j}$, the coordinate becomes constant and every label attains both ends.

For three distinct labels i, a, b, the distance identity gives

$$2r_i = d_1(y_i, y_a) + d_1(y_i, y_b) - d_1(y_a, y_b). \tag{2.1}$$

The triangle inequality shows that $r_i \ge 0$. Let $m_j = \min_k y_{kj}$ and $M_j = \max_k y_{kj}$. We claim that

$$M_j - y_{ij} \le r_i, \qquad y_{ij} - m_j \le r_i.$$

If $y_{ij} < M_j$, property (1) supplies distinct labels a, b attaining M_j, both different from i. The j-th coordinate contribution to the triangle excess in (2.1) is $2(M_j - y_{ij})$, and every other contribution is nonnegative. Thus $M_j - y_{ij} \le r_i$; when $y_{ij} = M_j$, this follows from $r_i \ge 0$. The minimum bound follows by reversing the coordinate. Since every y_{kj} lies in $[m_j, M_j]$, these inequalities imply $|y_{ij} - y_{kj}| \le r_i$. Interchanging i and k gives (4).

At most two labels move in each coordinate, so among the $2n+1$ labels there is an untouched label p, with $r_p = 1/2$. For $i \neq p$, the distance identity and (4) now yield

$$r_i + \frac{1}{2} = d_1(y_i, y_p) = \sum_{j=1}^{n} |y_{ij} - y_{pj}| \le n r_i.$$

Thus $r_i \ge 1/(2(n-1))$. The same bound holds for p, since $r_p = 1/2$ and $n \ge 2$, and the upper bound $r_i \le 1/2$ follows from $s_i \ge 0$.

□

The clipped points therefore carry a *star metric*: join each label i to a common centre by an edge of length r_i. The path distance between distinct leaves i, k is $r_i + r_k$, exactly $d_1(y_i, y_k)$. This is an abstract star; it does not assert the existence of a point $c \in \ell_1^n$ with $d_1(c, y_i) = r_i$ for every i.

3 Reciprocal weights and coordinate energies

Let $y_1, \dots, y_m \in \mathbb{R}^n$, with $m \ge 2$, and let $r_1, \dots, r_m > 0$ satisfy

$$d_1(y_i, y_\ell) = r_i + r_\ell \qquad (i \neq \ell).$$

We work throughout Sections 3 and 4 with an arbitrary positive star metric. The points need not arise from clipping, and coordinate extrema need not be attained more than once. Ordering the values in each coordinate expresses their distances as sums of intervening gaps, each associated with a cut in a nested family.

Definition 3.1 (Reciprocal masses, coordinate cuts, and energies) The *reciprocal masses* and their total are

$$\mu_i = \frac{1}{r_i}, \qquad T = \sum_{i=1}^{m} \mu_i.$$

For each coordinate j, choose a permutation π_j of the labels such that

$$y_{\pi_j(1),j} \le \cdots \le y_{\pi_j(m),j}.$$

For $1 \le k \le m-1$, define the *gaps* and *prefix cuts* by

$$\delta_{j,k} = y_{\pi_j(k+1),j} - y_{\pi_j(k),j}, \qquad C_{j,k} = \{\pi_j(1), \dots, \pi_j(k)\}.$$

The *cumulative masses* are

$$t_{j,0} = 0, \qquad t_{j,k} = \sum_{a=1}^{k} \mu_{\pi_j(a)} \quad (1 \le k \le m),$$

so $0 = t_{j,0} < \cdots < t_{j,m} = T$ and $t_{j,k} = \sum_{i \in C_{j,k}} \mu_i$ for $1 \le k \le m-1$. With

$$q_T(t) = \frac{t(T-t)}{T} \qquad (0 \le t \le T),$$

define the *span* and *energy* of coordinate j by

$$L_j = \sum_{k=1}^{m-1} \delta_{j,k} = \max_i y_{ij} - \min_i y_{ij}, \qquad E_j = \sum_{k=1}^{m-1} \delta_{j,k} q_T(t_{j,k}).$$

The cuts $C_{j,1} \subset \cdots \subset C_{j,m-1}$ form a nested family, with $C_{j,k}$ separating the first k labels in the coordinate ordering from the rest. A gap contributes to the distance between two coordinate values precisely when its cut separates their labels. Thus, writing $\mathbf{1}_C$ for the indicator of C, we have

$$|y_{ij} - y_{\ell j}| = \sum_{k=1}^{m-1} \delta_{j,k} |\mathbf{1}_{C_{j,k}}(i) - \mathbf{1}_{C_{j,k}}(\ell)|.$$

This is the nested cut representation used by Koolen, Laurent, and Schrijver [3]. Reordering labels within a block of tied values changes only cuts whose gaps are zero, so neither this representation nor the coordinate energies depend on the ordering of ties.

A cut with mass t on one side has mass $T - t$ on the other. Their product is the total weight of the separated pairs, where $\{i, \ell\}$ has weight $\mu_i \mu_\ell$. Thus $q_T(t)$ is this product divided by T, and E_j weights each coordinate gap by the normalized pair weight of its cut.

Proposition 3.2 (Exact total energy) *For each coordinate j,*

$$E_j = \frac{1}{T} \sum_{i<\ell} \mu_i \mu_\ell |y_{ij} - y_{\ell j}|.$$

The total energy satisfies

$$\sum_{j=1}^{n} E_j = m - 1.$$

Proof, The total weight of the unordered pairs separated by $C_{j,k}$ is

$$\sum_{i \in C_{j,k}} \sum_{\ell \notin C_{j,k}} \mu_i \mu_\ell = t_{j,k}(T - t_{j,k}).$$

Multiplying by $\delta_{j,k}$ and summing over the cuts counts each pair with the sum of the gaps separating its coordinate values. The nested cut representation therefore gives

$$T E_j = \sum_{i<\ell} \mu_i \mu_\ell |y_{ij} - y_{\ell j}|.$$

Summing over coordinates and using the star equations and $\mu_i r_i = 1$, we obtain

$$\begin{aligned} T \sum_{j=1}^{n} E_j &= \sum_{i<\ell} \mu_i \mu_\ell d_1(y_i, y_\ell) \\ &= \sum_{i<\ell} \mu_i \mu_\ell (r_i + r_\ell) = \sum_{i<\ell} (\mu_i + \mu_\ell) = (m-1)T. \end{aligned}$$

Dividing by $T > 0$ proves the total energy identity.

□

In particular,

$$\frac{1}{n}\sum_{j=1}^{n} E_j = 2 \qquad \text{when } m = 2n+1.$$

For clipped configurations in dimensions five and six, the subsequent estimates force this average strictly below 2, contradicting the identity.

To bound the individual coordinate energies, we use the following quadratic identity. Its proof is a weighted adaptation of the edge cut square decomposition used by Doust and Weston [9].

Lemma 3.3 (Weighted quadratic identity) *If $f_1, \dots, f_m \in \mathbb{R}$ satisfy*

$$\sum_{i=1}^{m} \mu_i f_i = 0,$$

then

$$\sum_{j=1}^{n}\sum_{k=1}^{m-1} \delta_{j,k}\left(\sum_{i\in C_{j,k}} \mu_i f_i\right)^2 = \sum_{i=1}^{m} \mu_i f_i^2.$$

Proof, Consider the signed pairwise sum

$$S = \sum_{i<\ell} \mu_i\mu_\ell f_i f_\ell d_1(y_i, y_\ell).$$

For any cut C, the zero-mean hypothesis gives

$$\sum_{i\in C}\sum_{\ell\notin C} \mu_i\mu_\ell f_i f_\ell = \left(\sum_{i\in C}\mu_i f_i\right)\left(\sum_{\ell\notin C}\mu_\ell f_\ell\right) = -\left(\sum_{i\in C}\mu_i f_i\right)^2.$$

Expanding the distances through the coordinate cuts consequently yields

$$S = -\sum_{j=1}^{n}\sum_{k=1}^{m-1} \delta_{j,k}\left(\sum_{i\in C_{j,k}} \mu_i f_i\right)^2.$$

On the other hand, the star equations give

$$\begin{aligned} S &= \sum_{i<\ell} \mu_i\mu_\ell f_i f_\ell (r_i + r_\ell) \\ &= \sum_{i=1}^{m} r_i\mu_i f_i \sum_{\ell\neq i} \mu_\ell f_\ell = -\sum_{i=1}^{m} r_i\mu_i^2 f_i^2 = -\sum_{i=1}^{m} \mu_i f_i^2, \end{aligned}$$

where we used $\sum_{\ell\neq i} \mu_\ell f_\ell = -\mu_i f_i$ and $r_i\mu_i = 1$. Comparing the two expressions for S proves the identity.

□

Every summand on the left-hand side of Lemma 3.3 is nonnegative, so discarding the contributions of all other coordinates gives an inequality for a single coordinate. We call a function $F : [0, T] \to \mathbb{R}$ *admissible* if it is continuous, piecewise C^1, and satisfies $F(0) = F(T) = 0$. For a fixed coordinate j, we choose the values f_i so that their weighted partial sums over $C_{j,k}$ equal $F(t_{j,k})$.

Lemma 3.4 (Coordinate test function inequality) *For each coordinate j and every admissible test function F,*

$$\sum_{k=1}^{m-1} \delta_{j,k} F(t_{j,k})^2 \le \sum_{k=1}^{m} \frac{[F(t_{j,k}) - F(t_{j,k-1})]^2}{t_{j,k} - t_{j,k-1}} \le \int_0^T F'(t)^2\, dt.$$

The final integral is the *Dirichlet energy* of F.

Proof, Fix j. Since $t_{j,k} - t_{j,k-1} = \mu_{\pi_j(k)} > 0$, we may assign to each label the slope over its mass interval:

$$f_{\pi_j(k)} = \frac{F(t_{j,k}) - F(t_{j,k-1})}{t_{j,k} - t_{j,k-1}} \qquad (1 \le k \le m).$$

Telescoping the weighted sums and using $F(0) = F(T) = 0$, we obtain

$$\sum_{i=1}^{m} \mu_i f_i = F(T) - F(0) = 0, \qquad \sum_{i \in C_{j,k}} \mu_i f_i = F(t_{j,k}) \quad (1 \le k \le m-1).$$

Applying Lemma 3.3 and retaining only the j-th coordinate gives the first inequality, because

$$\sum_{i=1}^{m} \mu_i f_i^2 = \sum_{k=1}^{m} \frac{[F(t_{j,k}) - F(t_{j,k-1})]^2}{t_{j,k} - t_{j,k-1}}.$$

On each mass interval, the fundamental theorem of calculus and Cauchy–Schwarz give

$$\frac{[F(t_{j,k}) - F(t_{j,k-1})]^2}{t_{j,k} - t_{j,k-1}} = \frac{\left(\int_{t_{j,k-1}}^{t_{j,k}} F'(t)\,dt\right)^2}{t_{j,k} - t_{j,k-1}} \le \int_{t_{j,k-1}}^{t_{j,k}} F'(t)^2\,dt.$$

Summing over $k = 1, \dots, m$ proves the second inequality.

□

4 Two bounds for a coordinate

We retain the notation and hypotheses of Section 3. Lemma 3.4 suggests taking $F = \sqrt{q_T}$, which would make its left-hand side equal to E_j. However, the derivative of $\sqrt{q_T}$ has inverse square root singularities at 0 and T, giving infinite Dirichlet energy. We therefore modify the function in two ways. In the span bound, F^2 approximates q_T with an additive error; summing this error over the gaps introduces a term proportional to the coordinate span. In the endpoint bound, F^2 agrees with q_T at every cumulative mass corresponding to a positive gap, and the Dirichlet energy of F is determined by the masses at the extreme coordinate levels.

4.1 The span bound

Proposition 4.1 (Span sensitive energy bound) *For every coordinate j and every $R > 1$,*

$$E_j \le a(R) + b(R) T L_j,$$

where

$$a(R) = \frac{R-1}{2(R+1)} \log R, \qquad b(R) = \frac{R}{(R+1)^2}.$$

Proof, Put

$$d = \frac{R-1}{R+1}, \qquad \lambda = \frac{T(1-d^2)}{4}, \qquad \tau = 2\lambda.$$

Since $0 < d < 1$, we have $0 < \tau < T/2$. The junctions $\tau = 2\lambda$ and $T - \tau$ are chosen so that the central branch $\sqrt{q_T - \lambda}$ meets linear terminal branches through $(0,0)$ and $(T,0)$ tangentially. Define

$$F(t) = \begin{cases} \dfrac{dt}{2\sqrt{\lambda}}, & 0 \le t \le \tau, \\ \sqrt{q_T(t) - \lambda}, & \tau \le t \le T - \tau, \\ \dfrac{d(T-t)}{2\sqrt{\lambda}}, & T - \tau \le t \le T. \end{cases}$$

Concavity of q_T gives $q_T(t) - \lambda \ge \lambda d^2 > 0$ on the central interval. At $t = \tau$, the central branch has value $d\sqrt{\lambda}$ and derivative $d/(2\sqrt{\lambda})$, matching the left linear branch. By symmetry, the branches also match in value and derivative at $T - \tau$. Thus F is admissible. On the central interval, $F^2 + \lambda = q_T$, while on $[0, \tau]$,

$$F(t)^2 + \lambda - q_T(t) = \frac{(t - 2\lambda)^2}{4\lambda} \ge 0.$$

Symmetry gives the same inequality on $[T-\tau, T]$, so $q_T \le F^2 + \lambda$ throughout $[0,T]$.

The two terminal branches contribute

$$\int_0^\tau F'(t)^2\,dt + \int_{T-\tau}^T F'(t)^2\,dt = 2\tau\frac{d^2}{4\lambda} = d^2.$$

On the central interval, the substitution $u = (2t/T - 1)/d$ gives

$$\begin{aligned}\int_\tau^{T-\tau} F'(t)^2\,dt &= \frac{d}{2}\int_{-d}^{d}\frac{u^2}{1-u^2}\,du \\ &= \frac{d}{2}\log\frac{1+d}{1-d} - d^2.\end{aligned}$$

Since $(1+d)/(1-d) = R$, it follows that

$$\int_0^T F'(t)^2\,dt = \frac{d}{2}\log R = a(R).$$

Applying the pointwise bound and Lemma 3.4, we obtain

$$\begin{aligned}E_j &\le \sum_{k=1}^{m-1}\delta_{j,k}F(t_{j,k})^2 + \lambda\sum_{k=1}^{m-1}\delta_{j,k} \\ &\le \int_0^T F'(t)^2\,dt + \lambda L_j = a(R) + b(R)TL_j,\end{aligned}$$

where $\lambda/T = (1-d^2)/4 = R/(R+1)^2 = b(R)$.

□

As R increases, $a(R)$ increases and $b(R)$ decreases. We defer the choice of R until Section 6.

4.2 The endpoint bound

A constant coordinate has $E_j = 0$, so only nonconstant coordinates need be considered.

Proposition 4.2 (Endpoint sensitive energy bound) *Suppose coordinate j is nonconstant, and let*

$$A_j = \sum_{i:\, y_{ij} = \min_\ell y_{\ell j}} \mu_i, \qquad B_j = \sum_{i:\, y_{ij} = \max_\ell y_{\ell j}} \mu_i$$

be the masses of its minimum and maximum groups. Then

$$E_j \le 1 + \frac{1}{4}\log\frac{(T-A_j)(T-B_j)}{A_jB_j}.$$

Proof, Write $A = A_j$ and $B = B_j$. Since the coordinate is nonconstant, the two extreme groups are disjoint and have positive masses, so $0 < A \le T - B < T$. Let F agree with $\sqrt{q_T}$ on $[A, T-B]$, with linear extensions to zero at 0 and T. More precisely,

$$F(t) = \begin{cases} \dfrac{t}{A}\sqrt{q_T(A)}, & 0 \le t \le A, \\ \sqrt{q_T(t)}, & A \le t \le T-B, \\ \dfrac{T-t}{B}\sqrt{q_T(T-B)}, & T-B \le t \le T. \end{cases}$$

The branches agree at their junctions, so F is admissible. Every positive gap lies after the entire minimum group and before the entire maximum group. Thus $\delta_{j,k} > 0$ implies $A \le t_{j,k} \le T - B$, where $F(t_{j,k})^2 = q_T(t_{j,k})$. Lemma 3.4 therefore gives

$$E_j = \sum_{k=1}^{m-1}\delta_{j,k}F(t_{j,k})^2 \le \int_0^T F'(t)^2\,dt.$$

The two terminal branches contribute

$$\int_0^A F'(t)^2\,dt + \int_{T-B}^T F'(t)^2\,dt = \frac{q_T(A)}{A} + \frac{q_T(T-B)}{B} = 2 - \frac{A+B}{T}.$$

On the central interval,

$$\left(\frac{d}{dt}\sqrt{q_T(t)}\right)^2 = \frac{(T-2t)^2}{4Tt(T-t)} = \frac{1}{4t} + \frac{1}{4(T-t)} - \frac{1}{T},$$

and hence

$$\begin{aligned}\int_A^{T-B} F'(t)^2\,dt &= \left[\frac{1}{4}\log\frac{t}{T-t} - \frac{t}{T}\right]_A^{T-B} \\ &= \frac{1}{4}\log\frac{(T-A)(T-B)}{AB} - \frac{T-A-B}{T}.\end{aligned}$$

Adding the terminal and central contributions yields

$$\int_0^T F'(t)^2\,dt = 1 + \frac{1}{4}\log\frac{(T-A)(T-B)}{AB}.$$

This calculation remains valid when $A+B=T$: the central interval collapses to a point, its integral is zero, the terminal contributions sum to 1, and the logarithm vanishes. Thus the bound holds in all cases, with $E_j \le 1$ when $A+B=T$.

□

We next use the clipping properties in Proposition 2.2 to bound the coordinate spans and endpoint masses appearing in these estimates.

5 Endpoint accounting and the scalar reduction

We now return to the clipped configurations of Section 2 and apply the general star metric estimates developed in Sections 3 and 4.

Throughout this section, fix a unit equilateral family $x_1, \dots, x_{2n+1} \in \ell_1^n$, where $n \ge 2$, and let $y_1, \dots, y_{2n+1}$ be its terminal clipping. We retain the earlier notation.

5.1 Mass accounting

Proposition 2.2 gives $2 \le \mu_i \le 2(n-1)$. If a label has positive excess mass $\mu_i - 2$, then $r_i < 1/2$, so it moved during clipping and attained a unique original coordinate end. Assigning its mass to each such end therefore counts every positive excess at least once. This motivates the following assignment.

Definition 5.1 (Assigned endpoint masses) An *original signed coordinate end* is a pair $e = (j, \sigma)$, where $1 \le j \le n$ and $\sigma \in \{-, +\}$. The sign $-$ denotes the minimum and $+$ the maximum of the original values $x_{1j}, \dots, x_{2n+1,j}$. Assign to each such end the mass

$$u_e = \begin{cases} \mu_i, & \text{if the end is attained uniquely at label } i, \\ 2, & \text{otherwise.} \end{cases}$$

Define

$$M = \max_i \mu_i, \qquad z = \frac{1}{2n}\sum_e u_e, \qquad w = \frac{z}{M}, \tag{5.1}$$

where the sum runs over all $2n$ original signed coordinate ends.

Since $2 \le u_e \le M$, we have

$$2 \le M \le 2(n-1), \qquad 2 \le z \le M, \qquad \frac{2}{M} \le w \le 1. \tag{5.2}$$

Lemma 5.2 (Endpoint accounting and span bounds) *The total mass satisfies*

$$T \le 2nz + 2. \tag{5.3}$$

For each original signed coordinate end e, the corresponding residual endpoint group has mass at least $u_e + 2$, and

$$u_e + 2 < \frac{T}{2}.$$

Moreover, for every coordinate j,

$$L_j \le \min\left\{\frac{1}{u_{j,-}}, \frac{1}{u_{j,+}}, \frac{2}{M}\right\}. \tag{5.4}$$

Proof, Each label with positive excess $\mu_i - 2$ contributes that excess at least once to $\sum_e (u_e - 2)$, and distinct labels contribute at distinct ends. Since all these terms are nonnegative,

$$T = 2(2n+1) + \sum_{i=1}^{2n+1} (\mu_i - 2) \le 2(2n+1) + \sum_e (u_e - 2) = 2nz + 2.$$

If an original end e is unique at label i, Proposition 2.2 ensures that i remains at the corresponding residual end and that at least one other label lies there. Since each label has mass at least 2, this group has mass at least $\mu_i + 2 = u_e + 2$. If the original end is not unique, at least two original endpoint labels remain in the residual group, giving mass at least $4 = u_e + 2$. This also covers constant coordinates, for which the residual group at either end contains every label. Furthermore,

$$u_e + 2 \le M + 2 \le 2n < 2n + 1 \le \frac{T}{2},$$

where the last inequality follows from $\mu_i \ge 2$.

It remains to bound the spans. If an original end (j, σ) is unique at label i, then i remains at that residual end. Choose a label k at the opposite residual end. The coordinate bound in Proposition 2.2 gives

$$L_j = |y_{ij} - y_{kj}| \le r_i = \frac{1}{u_{j,\sigma}}.$$

If the original end is not unique, choosing labels at the two residual ends gives $L_j \le 1/2 = 1/u_{j,\sigma}$ by the same coordinate bound and $r_i \le 1/2$. Finally, choose a label i_* with $\mu_{i_*} = M$, and labels a, b at the residual minimum and maximum in coordinate j. Applying the coordinate bound to (a, i_*) and (i_*, b) yields

$$L_j = (y_{i_* j} - y_{aj}) + (y_{bj} - y_{i_* j}) \le 2r_{i_*} = \frac{2}{M}.$$

□

5.2 Averaging the span bound

To average the normalized endpoint span bounds using only the mean normalized endpoint mass w, we seek a concave majorant of $\min\{2, 1/v\}$. Define

$$h(v) = \min\{2, 3 - 2v\} \qquad (0 < v \le 1).$$

The line $3 - 2v$ is the chord joining $(1/2, 2)$ and $(1, 1)$ on the graph of the convex function $1/v$, so it lies above $1/v$ on $[1/2, 1]$. On $(0, 1/2]$, both $h(v)$ and $\min\{2, 1/v\}$ equal 2. Since the minimum of two affine functions is concave, h is a concave majorant of $\min\{2, 1/v\}$ on $(0, 1]$.

Proposition 5.3 (Average span bound) *Set*

$$Q = \frac{T}{n} \sum_{j=1}^{n} L_j, \qquad \Phi_n(M, w) = \left(2nw + \frac{2}{M}\right) h(w). \tag{5.5}$$

Then

$$Q \le \Phi_n(M, w) \le \frac{(3n + 2/M)^2}{4n}. \tag{5.6}$$

Proof, For each end $e = (j, \sigma)$, Lemma 5.2 and the majorant property give

$$ML_j \le \min\left\{2, \frac{M}{u_e}\right\} \le h\left(\frac{u_e}{M}\right).$$

Averaging over the $2n$ ends counts each span twice. By concavity of h,

$$\frac{M}{n} \sum_{j=1}^{n} L_j \le \frac{1}{2n} \sum_e h\left(\frac{u_e}{M}\right) \le h\left(\frac{1}{2n} \sum_e \frac{u_e}{M}\right) = h(w).$$

Since $h(w) > 0$, the mass accounting inequality in Lemma 5.2 and $z = Mw$ yield

$$Q \le \frac{T}{M} h(w) \le \left(2nw + \frac{2}{M}\right) h(w) = \Phi_n(M, w).$$

Finally, using $h(w) \le 3 - 2w$ and completing the square, we obtain

$$\begin{aligned}\Phi_n(M, w) &\le \left(2nw + \frac{2}{M}\right)(3 - 2w) \\ &= \frac{(3n + 2/M)^2}{4n} - 4n\left(w - \frac{3n - 2/M}{4n}\right)^2 \le \frac{(3n + 2/M)^2}{4n}.\end{aligned}$$

□

5.3 Averaging the endpoint bound

Since the endpoint energy estimate subtracts the logarithms of the endpoint masses, an upper bound for the average energy requires a lower bound for the average of these logarithms. The logarithmic chord provides an affine lower bound whose average, for fixed M, depends only on z; define its slope by

$$\alpha(M) = \begin{cases} 1/4, & M = 2, \\ \dfrac{\log((M+2)/4)}{M-2}, & M > 2. \end{cases} \tag{5.7}$$

For $M > 2$, this is the secant slope of the logarithm between 4 and $M + 2$; the value at $M = 2$ is the limiting slope.

Lemma 5.4 (Logarithmic chord bound) *The function α is positive, nonincreasing, and at most $1/4$ on $[2, \infty)$. For $2 \le u \le M$,*

$$\log(u + 2) \ge \log 4 + (u - 2)\alpha(M).$$

Proof, Since the logarithm is strictly increasing, its secant slopes are positive. By concavity, the secant slope from the fixed left endpoint 4 is nonincreasing as the right endpoint increases, and is bounded above by $(\log)'(4) = 1/4$. Its limit as the right endpoint decreases to 4 is $1/4$, so the definition at $M = 2$ preserves these properties.

For $M > 2$, concavity also gives

$$\begin{aligned}\log(u + 2) &\ge \frac{M - u}{M - 2} \log 4 + \frac{u - 2}{M - 2} \log(M + 2) \\ &= \log 4 + (u - 2)\alpha(M).\end{aligned}$$

When $M = 2$, necessarily $u = 2$, and the inequality is an equality.

□

Proposition 5.5 (Average endpoint bound) *Define*

$$G_n(M, z) = \log \frac{(2n - 1)z}{4} - (z - 2)\alpha(M). \tag{5.8}$$

Then

$$\frac{1}{n} \sum_{j=1}^{n} E_j \le 1 + \frac{1}{2} G_n(M, z). \tag{5.9}$$

Proof, For a nonconstant coordinate j, Lemma 5.2 gives $A_j \ge u_{j,-} + 2$ and $B_j \ge u_{j,+} + 2$. Since $t \mapsto \log((T - t)/t)$ is decreasing on $(0, T)$, Proposition 4.2 yields

$$E_j \le 1 + \frac{1}{4} \sum_{\sigma \in \{-,+\}} \log \frac{T - u_{j,\sigma} - 2}{u_{j,\sigma} + 2}. \tag{5.10}$$

The same inequality holds for a constant coordinate: $E_j = 0$, whereas both logarithms are positive because $u_{j,\sigma} + 2 < T/2$ by Lemma 5.2. Thus averaging (5.10) over all coordinates gives

$$\frac{1}{n}\sum_{j=1}^{n} E_j \le 1 + \frac{1}{4n}\sum_{e}\big[\log(T - u_e - 2) - \log(u_e + 2)\big].$$

All numerator arguments are positive. Jensen's inequality for the logarithm, followed by the mass accounting inequality in Lemma 5.2, gives

$$\frac{1}{2n}\sum_{e}\log(T - u_e - 2) \le \log(T - z - 2) \le \log\big((2n-1)z\big).$$

For the denominator logarithms, Lemma 5.4 and $2 \le u_e \le M$ give

$$\frac{1}{2n}\sum_{e}\log(u_e + 2) \ge \log 4 + \frac{\alpha(M)}{2n}\sum_{e}(u_e - 2) = \log 4 + (z-2)\alpha(M).$$

Combining these estimates, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} E_j \le 1 + \frac{1}{2}\left[\log\frac{(2n-1)z}{4} - (z-2)\alpha(M)\right] = 1 + \frac{1}{2}G_n(M,z).$$

□

5.4 The necessary scalar inequalities

Corollary 5.6 (Scalar obstruction) *Every hypothetical $(2n+1)$-point unit equilateral family in ℓ_1^n, with $n \ge 2$, produces an admissible pair $(M, w) \in \mathcal{D}_n$, where*

$$\mathcal{D}_n = \left\{(M,w) : 2 \le M \le 2(n-1),\ \frac{2}{M} \le w \le 1\right\}, \tag{5.11}$$

satisfying

$$2 \le a(R) + b(R)\Phi_n(M,w) \qquad \text{for every } R > 1, \tag{5.12}$$

and

$$2 \le 1 + \frac{1}{2}G_n(M, Mw). \tag{5.13}$$

Here

$$\Phi_n(M,w) = \left(2nw + \frac{2}{M}\right)h(w), \qquad G_n(M,z) = \log\frac{(2n-1)z}{4} - (z-2)\alpha(M).$$

Proof, The ranges in (5.2) give $(M,w) \in \mathcal{D}_n$. By Proposition 3.2, the average coordinate energy is 2. Averaging the bound in Proposition 4.1 and applying Proposition 5.3, with $b(R) > 0$, gives

$$2 = \frac{1}{n}\sum_{j=1}^{n} E_j \le a(R) + b(R)Q \le a(R) + b(R)\Phi_n(M,w) \qquad (R > 1).$$

The endpoint inequality (5.13) follows from Proposition 5.5 and $z = Mw$.

□

To prove Theorem 1.1, it remains to show that these necessary inequalities have no simultaneous solution in $\mathcal{D}_n$ for $n = 5, 6$.

5.5 Monotonicity in the dimension

Increasing the ambient dimension enlarges the admissible parameter region and weakens both upper bounds. Consequently, if the necessary inequalities have no simultaneous solution in one dimension, the same holds in every smaller dimension covered by the reduction.

Lemma 5.7 (Dimension monotonicity) *For $n \geq 2$,*

$$\mathcal{D}_n \subseteq \mathcal{D}_{n+1}.$$

Moreover, for every $(M, w) \in \mathcal{D}_n$,

$$\Phi_{n+1}(M, w) - \Phi_n(M, w) = 2w\, h(w) > 0,$$
$$G_{n+1}(M, Mw) - G_n(M, Mw) = \log \frac{2n+1}{2n-1} > 0.$$

Hence, for every $R > 1$,

$$a(R) + b(R)\Phi_n(M, w) < a(R) + b(R)\Phi_{n+1}(M, w),$$
$$1 + \frac{1}{2}G_n(M, Mw) < 1 + \frac{1}{2}G_{n+1}(M, Mw).$$

Proof, The inclusion follows from $2(n-1) < 2n$, since the other constraints defining $\mathcal{D}_n$ do not depend on n. Subtracting the expressions for Φ_{n+1} and Φ_n gives the first difference. In $G_n(M, Mw)$, only the factor $2n-1$ inside the logarithm depends on n, giving the second. Since $0 < w \leq 1$, we have $h(w) \geq 1$; thus both differences are positive, and $b(R) > 0$ gives the stated strict inequalities.

□

Corollary 5.8 (Exclusion in lower dimensions) *Suppose that, for some $N \geq 2$, every point of $\mathcal{D}_N$ violates at least one of the scalar necessary inequalities (5.12) and (5.13). Then the same is true of every point of $\mathcal{D}_n$ for $2 \leq n \leq N$. In particular, to prove Theorem 1.1, it is enough to show that the necessary inequalities have no simultaneous solution in $\mathcal{D}_6$.*

Proof, For $2 \leq n \leq N$, iterating Lemma 5.7 gives $\mathcal{D}_n \subseteq \mathcal{D}_N$, and neither right-hand side decreases when n is replaced by N. Thus any point satisfying both necessary inequalities in dimension n, with the span inequality holding for every $R > 1$, would also satisfy them in dimension N, contrary to the hypothesis. The last assertion follows from Corollary 5.6.

□

6 Scalar exclusion in dimension six

We show that no point of $\mathcal{D}_6$ satisfies both necessary inequalities of Corollary 5.6. The exclusion in dimension five then follows from Corollary 5.8.

The scalar reduction lets us exclude all of $\mathcal{D}_6$ without determining which pairs are geometrically realizable. At each pair, it suffices to force one of the two scalar upper bounds below the exact average energy 2.

Throughout this section, we specialize the span estimate to $R = 20$. This choice is convenient for the estimates below; no optimality is needed. Since $\log 20 < 3$, the coefficients in Proposition 4.1 satisfy

$$a(20) = \frac{19}{42}\log 20 < \frac{19}{14}, \qquad b(20) = \frac{20}{441}. \tag{6.1}$$

The span bound suffices when M or w is sufficiently large; the endpoint bound covers the remaining region.

6.1 The critical span threshold

The span bound excludes any pair with $\Phi_6(M, w) \leq Q_*$, where

$$Q_* = \frac{3969}{280}, \qquad \frac{19}{14} + \frac{20}{441}Q_* = 2. \tag{6.2}$$

If $\Phi_6(M,w) \le Q_*$, then (6.1) gives

$$a(20) + b(20)\Phi_6(M,w) < \frac{19}{14} + \frac{20}{441}Q_* = 2, \tag{6.3}$$

contrary to the necessary scalar inequality (5.12). Equality at $\Phi_6(M,w) = Q_*$ suffices because $a(20) < 19/14$ is strict.

The global bound in Proposition 5.3 is

$$\Phi_6(M,w) \le \frac{(18 + 2/M)^2}{24}.$$

This envelope decreases with M and equals Q_* at

$$M_{\mathrm{span}} = \frac{20 + 2\sqrt{105}}{9} \approx 4.4993.$$

Since $\sqrt{105} < 41/4$, we have $M_{\mathrm{span}} < 9/2$. Thus $9/2$ is a convenient rational cutoff: for $M \ge 9/2$, the global bound gives $\Phi_6(M,w) < Q_*$.

The cutoff $M = 4$ comes from the endpoint bound. Write $z = Mw$. By (5.8) and Lemma 5.4, for $2 \le z \le M \le 4$,

$$\frac{\partial G_6}{\partial z}(M,z) = \frac{1}{z} - \alpha(M) \ge \frac{1}{M} - \frac{1}{4} \ge 0.$$

Hence $G_6(M,z) \le G_6(M,M)$ throughout this region, so the endpoint bound reduces to its diagonal value.

In the transition range $4 < M < 9/2$, retaining w allows the two bounds to complement one another. The restriction $w \le 4/5$ gives $z = Mw < 18/5$, allowing a sharper endpoint comparison than replacing z by M. For larger w, the decreasing branch of the span bound becomes effective.

Since $h(w) \le 3 - 2w$ and $2/M < 1/2$,

$$\Phi_6(M,w) \le \left(12w + \frac{2}{M}\right)(3 - 2w) < \left(12w + \frac{1}{2}\right)(3 - 2w).$$

Writing $f(w) = (12w + 1/2)(3 - 2w)$, we have $f'(w) = 35 - 48w$. On the decreasing branch $w > 35/48$, the equation $f(w) = Q_*$ has the solution

$$w_* = \frac{35}{48} + \frac{\sqrt{205}}{240} \approx 0.7888.$$

Since $\sqrt{205} < 17$, we have $w_* < 4/5$. Thus, if $w > 4/5$, then $\Phi_6(M,w) < f(w) < Q_*$, so the span bound suffices throughout the transition range.

We shall use the two bounds in the following regions:

$2 \le M \le 4$		endpoint bound,
$4 < M < 9/2$,	$w \le 4/5$	endpoint bound,
$4 < M < 9/2$,	$w > 4/5$	span bound,
$M \ge 9/2$		span bound.

6.2 Exclusion of the scalar region in dimension six

We now apply the span and endpoint bounds in the four regions identified above to show that the scalar necessary inequalities cannot hold simultaneously.

Proposition 6.1 (Scalar exclusion in dimension six) *For every $(M, w) \in \mathcal{D}_6$, at least one of the inequalities*

$$a(20) + b(20)\Phi_6(M, w) < 2 \qquad \text{or} \qquad 1 + \frac{1}{2}G_6(M, Mw) < 2$$

holds. Consequently, no point of $\mathcal{D}_6$ satisfies the necessary scalar conditions of Corollary 5.6.

Proof, Fix $(M, w) \in \mathcal{D}_6$ and set $z = Mw$, so that $2 \le z \le M$. For the endpoint inequality, it suffices to show $G_6(M, z) < 2$; for the span inequality, it suffices to show $\Phi_6(M, w) \le Q_*$, by (6.3).

Case 1: $2 \le M \le 4$. By the monotonicity in z established in Subsection 6.1 and the fact that $M \mapsto 11M/(M + 2)$ is increasing,

$$G_6(M, z) \le G_6(M, M) = \log \frac{11M}{M + 2} \le \log \frac{22}{3} < 2,$$

where the diagonal identity also holds at $M = 2$.

Case 2: $4 < M < 9/2$ *and* $w \le 4/5$. Here $z < 18/5$. By Lemma 5.4, α is nonincreasing and at most $1/4$. For fixed $z \ge 2$, $G_6(M, z)$ is therefore nondecreasing in M. Moreover, for $2 \le z \le 18/5$,

$$\frac{\partial G_6}{\partial z}\left(\frac{9}{2}, z\right) = \frac{1}{z} - \alpha\left(\frac{9}{2}\right) \ge \frac{5}{18} - \frac{1}{4} > 0.$$

It follows that

$$\begin{aligned} G_6(M, z) &< G_6\left(\frac{9}{2}, \frac{18}{5}\right) \\ &= \log \frac{99}{10} - \frac{16}{25} \log \frac{13}{8} < 2. \end{aligned}$$

Case 3: $4 < M < 9/2$ *and* $w > 4/5$. Since $2/M < 1/2$, Proposition 5.3 gives $\Phi_6(M, w) < f(w)$, with f as defined in Subsection 6.1. The function f is decreasing on $[4/5, 1]$, so

$$\Phi_6(M, w) < f(w) < f\left(\frac{4}{5}\right) < Q_*.$$

Case 4: $M \ge 9/2$. Now $2/M \le 4/9$, and the global bound in Proposition 5.3 yields

$$\Phi_6(M, w) \le \frac{(18 + 4/9)^2}{24} < Q_*.$$

The four cases exhaust $\mathcal{D}_6$, proving the proposition.

□

6.3 Completion of the main theorem

Together with the reduction in Section 5, the scalar exclusion rules out the corresponding unit equilateral families in dimensions five and six.

Corollary 6.2 (Exclusion in dimensions five and six) *There is no eleven-point unit equilateral family in ℓ_1^5, and there is no thirteen-point unit equilateral family in ℓ_1^6.*

Proof, By Corollary 5.6, a thirteen-point unit equilateral family in ℓ_1^6 would determine a pair $(M, w) \in \mathcal{D}_6$ satisfying both necessary scalar inequalities. Proposition 6.1 shows that every such pair violates at least one of them, so no such family exists.

Since $\mathcal{D}_5 \subseteq \mathcal{D}_6$ and the scalar upper bounds are no larger in dimension five, Corollary 5.8 transfers the scalar exclusion to $\mathcal{D}_5$. Applying Corollary 5.6 with $n = 5$ therefore rules out an eleven-point unit equilateral family in ℓ_1^5.

□

Theorem 1.1 follows by scaling and the standard coordinate construction.

Proof of Theorem 1.1, Let $E \subset \ell_1^n$ be an equilateral family, where $n = 5$ or $n = 6$. If $|E| > 2n$, scale its common distance to 1 and choose a subset of $2n+1$ points, contrary to Corollary 6.2. Hence

$$e(\ell_1^5) \le 10, \qquad e(\ell_1^6) \le 12.$$

Conversely, the signed coordinate vectors

$$\{\pm\tfrac{1}{2}e_j : 1 \le j \le n\}$$

form a $2n$-point equilateral set in ℓ_1^n, with common distance one. Thus $e(\ell_1^n) \ge 2n$, and combining the upper and lower bounds gives

$$e(\ell_1^5) = 10, \qquad e(\ell_1^6) = 12.$$

□

Remark 6.3 (A direct proof in dimension five) The proof in dimension six uses the endpoint estimate to cover the regions left by the span bound. In dimension five, the uniform span bound alone excludes the entire admissible region $\mathcal{D}_5$: since $M \ge 2$, Proposition 5.3 gives

$$Q \le \Phi_5(M, w) \le \frac{(15 + 2/M)^2}{20} \le \frac{(15+1)^2}{20} = \frac{64}{5}.$$

Thus the necessary scalar inequality (5.12) with $n = 5$ and $R = 20$ would imply

$$\begin{aligned} 2 &\le a(20) + \frac{20}{441}\Phi_5(M, w) \\ &< \frac{19}{14} + \frac{20}{441}\frac{64}{5} = \frac{1709}{882} < 2, \end{aligned}$$

a contradiction. The span estimate alone therefore rules out eleven equilateral points in ℓ_1^5.

Thus the uniform envelope lies below Q_* throughout $\mathcal{D}_5$. In dimension six, the endpoint estimate covers small M and, in the transition range, small w.

7 Concluding remarks

7.1 Formal verification

The equalities $e(\ell_1^5) = 10$ and $e(\ell_1^6) = 12$ have been formalized in Lean using mathlib. The formalization covers the geometric reduction, the analytic estimates, and the scalar exclusion, together with the signed coordinate construction giving the lower bounds. The source code is available in the accompanying repository.[1]

7.2 The limit of the scalar reduction

We conclude by identifying the obstruction to extending the preceding scalar argument to dimension seven.

In dimension six, the span and endpoint estimates complement one another: at every point of $\mathcal{D}_6$, at least one bounds the average energy strictly below 2. In dimension seven, the two necessary inequalities have simultaneous solutions in $\mathcal{D}_7$, even when the span estimate is optimized over all $R > 1$.

Proposition 7.1 (A surviving scalar pair in dimension seven) *The pair* $(M, w) = (4, 3/4)$ *belongs to* $\mathcal{D}_7$ *and satisfies*

$$a(R) + b(R)\Phi_7\left(4, \frac{3}{4}\right) > 2 \qquad \textit{for every } R > 1,$$

as well as

$$1 + \frac{1}{2}G_7(4, 3) > 2.$$

[1] https://github.com/kusner-l1/kusnerl1-lean

Proof, The pair $(4, 3/4)$ is admissible, and $h(3/4) = 3/2$ gives

$$\Phi_7\left(4, \frac{3}{4}\right) = \frac{33}{2}.$$

For the span estimate, put $x = (R-1)/(R+1) \in (0,1)$. Then $\log R = 2\operatorname{arctanh} x$ and $b(R) = (1-x^2)/4$, so

$$a(R) + \frac{33}{2} b(R) = x \operatorname{arctanh} x + \frac{33}{8}(1-x^2).$$

Writing $y = x^2$ and retaining the first nine terms of the positive series

$$x \operatorname{arctanh} x = \sum_{k=1}^{\infty} \frac{y^k}{2k-1},$$

we obtain

$$a(R) + \frac{33}{2} b(R) - 2 \geq P(y), \qquad P(y) = \frac{17}{8} - \frac{25}{8} y + \sum_{k=2}^{9} \frac{y^k}{2k-1}.$$

The polynomial P is strictly convex on $[0,1]$, since $P''(y) \geq 2/3 > 0$ there. Direct calculation gives

$$P'\left(\frac{9}{10}\right) < 0, \qquad P\left(\frac{9}{10}\right) + \frac{1}{10} P'\left(\frac{9}{10}\right) > \frac{3}{400}.$$

Thus the tangent to P at $9/10$ is decreasing and has value greater than $3/400$ at 1. By convexity,

$$P(y) \geq P\left(\frac{9}{10}\right) + P'\left(\frac{9}{10}\right)\left(y - \frac{9}{10}\right) > \frac{3}{400} \qquad (0 \leq y \leq 1).$$

It follows that $a(R) + (33/2) b(R) > 2 + 3/400$ for every $R > 1$.

For the endpoint estimate, $z = Mw = 3$ and $\alpha(4) = \frac{1}{2}\log(3/2)$, giving

$$G_7(4,3) = \frac{1}{2} \log \frac{507}{8} > 2.$$

□

At the same pair in dimension six,

$$\Phi_6\left(4, \frac{3}{4}\right) = \frac{57}{4}, \qquad G_6(4,3) = \log \frac{33}{4} - \frac{1}{2} \log \frac{3}{2} < 2,$$

so the endpoint estimate excludes the pair. Passing to dimension seven increases $\Phi_n(4, 3/4)$ by $9/4$ and $G_n(4,3)$ by $\log(13/11)$, taking $G_7(4,3)$ above its critical value 2.

The uniform margin in the proof, together with $0 < b(R) \leq 1/4$ and continuity of Φ_7 and G_7, shows that both strict inequalities persist in a neighborhood of $(4, 3/4)$ contained in $\mathcal{D}_7$. By Lemma 5.7, the same pair also satisfies both necessary inequalities for every $n \geq 7$.

The existing inequalities admit scalar data in dimension seven, but this does not establish their geometric realizability, nor does it rule out sharper estimates in the same parameters.

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