A characterization of q-monotonicity by Bernstein polynomial inequalities

Drew Gallinger

University of Toronto, Canada.

Corresponding author(s). E-mail(s): drew.gallinger@utoronto.ca;

Abstract

We prove a converse to a Bernstein polynomial inequality for q-monotone functions. Let $q \geq 2$ and $f \in C[0,1]$. If f satisfies the strengthened inequality of Abel, Leviatan, and Raşa (*Results Math.* **77**, 239, 2022) for every $n \geq 1$ and $0 \leq y < x \leq 1$, then f is q-monotone. This resolves their open problem under continuity alone, removing the smoothness assumptions in their partial converses (*Results Math.* **78**, 41, 2023). Together with the known direct implication, this yields a characterization of q-monotonicity in $C[0,1]$ in terms of Bernstein polynomial inequalities.

Keywords: Bernstein polynomials, q-monotonicity, polynomial inequalities, functional inequalities

MSC Classification (2020): Primary 26D05 , 39B62; Secondary 41A17 , 41A36

1 Introduction and Main Result

We begin by recalling the required notation and known results for Bernstein polynomials. Let $C[0,1]$ denote the space of continuous real-valued functions on $[0,1]$.

Definition 1.1 (Bernstein polynomials) For an integer $m \geq 1$ and $0 \leq k \leq m$, the Bernstein basis polynomials are defined as

$$p_{m,k}(t) = \binom{m}{k} t^k (1-t)^{m-k}, \quad t \in [0,1].$$

We set $p_{0,0}(t) = 1$. The mth *Bernstein polynomial* of $f \in C[0,1]$ is defined by

$$(B_m f)(t) = \sum_{k=0}^{m} p_{m,k}(t) f\left(\frac{k}{m}\right). \tag{1.1}$$

It is known that the polynomials $B_m f$ converge uniformly to f on $[0,1]$. Moreover, Bernstein polynomials preserve monotonicity and convexity, that is, if f is nondecreasing or convex, then so is $B_m f$ for every m [7]. To describe the higher-order property of q-monotonicity, we use forward differences.

Definition 1.2 (Forward differences and q-monotonicity) Let $q \geq 1$ be an integer and let $f \in C[0,1]$. The qth *forward difference* of f with step $h > 0$ is

$$\Delta_h^q f(u) = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} f(u + kh), \tag{1.2}$$

where $u \geq 0$ and $u + qh \leq 1$. The function f is *q-monotone* on $[0,1]$ if

$$\Delta_h^q f(u) \geq 0$$

for every such u and h [2].

The cases $q = 1$ and $q = 2$ correspond to nondecreasing and convex functions, respectively. Moreover if $f \in C^q[0,1]$, then f is q-monotone if and only if $f^{(q)}(u) \geq 0$ for every $u \in [0,1]$. To formulate the Bernstein polynomial inequality studied in this paper, we introduce the following mixed sums and their associated comparison gap.

Definition 1.3 (Mixed Bernstein sums and comparison gap) Let $n \geq 1$ and $q \geq 2$ be integers, let $f \in C[0,1]$, and let $x, y \in [0,1]$. For $0 \leq j \leq q$, the *mixed Bernstein sum* is

$$(T_{n,q,j} f)(x,y) = \sum_{a=0}^{jn} \sum_{b=0}^{(q-j)n} p_{jn,a}(x)\, p_{(q-j)n,b}(y)\, f\left(\frac{a+b}{qn}\right). \tag{1.3}$$

The associated *comparison gap* is

$$H_{n,q} f(x,y) = \sum_{j=1}^{q-1} (-1)^{q-j+1} \binom{q}{j} \left[(B_{qn} f)\left(\frac{jx + (q-j)y}{q}\right) - (T_{n,q,j} f)(x,y) \right]. \tag{1.4}$$

The notation in (1.3) compresses the q degree-n Bernstein sums occurring in the original formulation. Indeed, repeated use of Vandermonde's identity [5] gives, for $1 \leq j \leq q-1$,

$$\sum_{\substack{0 \leq \nu_1, \ldots, \nu_j \leq n \\ \nu_1 + \cdots + \nu_j = a}} \prod_{i=1}^{j} p_{n,\nu_i}(x) = p_{jn,a}(x),$$

and analogously for the remaining $q - j$ factors at y. Consequently,

$$(T_{n,q,j} f)(x,y) = \sum_{\nu_1, \ldots, \nu_q = 0}^{n} \left(\prod_{i=1}^{j} p_{n,\nu_i}(x) \right) \left(\prod_{i=j+1}^{q} p_{n,\nu_i}(y) \right) \times f\left(\frac{\nu_1 + \cdots + \nu_q}{qn}\right).$$

Thus (1.4) is exactly the strengthened comparison of Abel, Leviatan, and Raşa, written in compressed notation. At the endpoints,

$$(T_{n,q,0} f)(x,y) = (B_{qn} f)(y), \qquad (T_{n,q,q} f)(x,y) = (B_{qn} f)(x).$$

Abel, Leviatan, and Raşa [2] proved that every q-monotone $C[0,1]$ function, $q \geq 3$, satisfies

$$H_{n,q} f(x,y) \geq 0 \qquad (n \geq 1, \quad 0 \leq y < x \leq 1). \tag{1.5}$$

For $q = 2$, this reduces to

$$(B_{2n} f)\left(\frac{x+y}{2}\right) \geq \sum_{a=0}^{n} \sum_{b=0}^{n} p_{n,a}(x)\, p_{n,b}(y)\, f\left(\frac{a+b}{2n}\right),$$

the sharpened inequality for convex functions proved by Komisarski and Rajba [6] and by Abel and Raşa [4].

The gap satisfies $H_{n,q}f(y,x) = (-1)^q H_{n,q}f(x,y)$ and vanishes when $x = y$, as follows by reindexing (1.4). Hence (1.5) is equivalent to the all pairs formulation $\operatorname{sgn}(x-y)^q H_{n,q}f(x,y) \geq 0$ used in [2].

To relate (1.5) to the earlier inequality, use the endpoint identities $T_{n,q,0}f(x,y) = (B_{qn}f)(y)$ and $T_{n,q,q}f(x,y) = (B_{qn}f)(x)$. Collecting the remaining Bernstein terms into a forward difference gives

$$\sum_{j=0}^{q}(-1)^{q-j}\binom{q}{j}(T_{n,q,j}f)(x,y) = H_{n,q}f(x,y) + \Delta^q_{(x-y)/q}(B_{qn}f)(y), \qquad x > y. \quad (1.6)$$

Abel and Leviatan [1] proved that the sum on the left is nonnegative for q-monotone functions. Since Bernstein polynomials preserve q-monotonicity, the forward-difference term on the right is also nonnegative for such functions. Consequently, (1.5) improves the earlier lower bound from zero to the nonnegative quantity $\Delta^q_{(x-y)/q}(B_{qn}f)(y)$.

Abel, Leviatan, and Raşa [2] proved the converse of the earlier inequality for continuous functions. They then asked whether (1.5), assumed for every $n \geq 1$, also forces q-monotonicity under continuity alone. The earlier converse does not answer this question: for an arbitrary continuous f, the last term in (1.6) has no known sign.

For $q = 2$, the same paper gives a partial answer when f is twice differentiable in $(0,1)$ [2]. In their follow-up, Abel, Leviatan, and Raşa [3] proved the converse for $q \geq 4$ under the same differentiability assumption. For $q = 3$, they assumed that f' is absolutely continuous on every closed subinterval of $(0,1)$. These partial converses allow the inequality to hold along a subsequence of degrees chosen separately for each pair (x,y). Their proofs use Bernstein asymptotics.

Our main result gives a uniform characterization for every integer $q \geq 2$ under continuity alone.

Theorem 1.1 (Characterization of q-monotonicity) *Let $q \geq 2$ be an integer and let $f \in C[0,1]$. Then f is q-monotone on $[0,1]$ if and only if*

$$H_{n,q}f(x,y) \geq 0$$

for every integer $n \geq 1$ and every $0 \leq y < x \leq 1$.

The forward implication is the known inequality stated above; the converse answers the open question in [2].

The proof fixes n and, for $u \in (0,1)$, expands $H_{n,q}f(u+\varepsilon, u)$ about $\varepsilon = 0$. The coefficients of ε^r vanish for $0 \leq r < q$, while the coefficient of ε^q is a strictly positive multiple of $(B_{qn}f)^{(q)}(u)$. The assumed inequality therefore implies that $(B_{qn}f)^{(q)}$ is nonnegative on $(0,1)$, and continuity extends this conclusion to $[0,1]$. Hence $B_{qn}f$ has nonnegative qth forward differences. Passing to the limit using uniform convergence shows that f is q-monotone. Only the Bernstein polynomials are differentiated; f itself need only be continuous.

2 Proof of the Main Theorem

We prove the converse by computing the first nonzero coefficient of the comparison gap along the diagonal. For a polynomial P in an auxiliary variable z, write $[z^k]P(z)$ for the coefficient of z^k. The generating identity

$$\sum_{k=0}^{m} p_{m,k}(t)z^k = (1 - t + tz)^m \quad (2.1)$$

will allow us to carry out the computation before applying the sampled values of f.

Lemma 2.1 (Diagonal coefficient identity) *Let $q \geq 2$ and $n \geq 1$ be integers, let $f \in C[0,1]$, and put $N = qn$. For $u \in (0,1)$,*

$$\lim_{\varepsilon \downarrow 0} \frac{H_{n,q} f(u+\varepsilon, u)}{\varepsilon^q} = \left(\frac{n^q}{(N)_q} - \frac{1}{q^q} \right) (B_N f)^{(q)}(u), \tag{2.2}$$

where $(N)_q = N(N-1)\cdots(N-q+1)$. The coefficient multiplying $(B_N f)^{(q)}(u)$ is strictly positive.

Proof Set

$$A(z) = 1 - u + uz, \qquad b(z) = z - 1,$$

and define, on polynomials of degree at most N,

$$\mathcal{L}(P) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) [z^k] P(z).$$

It follows from (2.1) that, for $0 \leq j \leq q$,

$$(T_{n,q,j} f)(u+\varepsilon, u) = \mathcal{L}\Big((A + \varepsilon b)^{jn} A^{(q-j)n} \Big),$$
$$(B_N f)\left(u + \frac{j}{q}\varepsilon\right) = \mathcal{L}\left(\left(A + \frac{j}{q}\varepsilon b \right)^N \right).$$

Let $w_j = (-1)^{q-j}\binom{q}{j}$. At $j = 0$ and $j = q$, the mixed sum and the corresponding Bernstein polynomial agree. These zero terms may therefore be inserted into (1.4). Reversing the difference in brackets changes its coefficient from $(-1)^{q-j+1}\binom{q}{j}$ to w_j. The binomial theorem then gives

$$H_{n,q} f(u+\varepsilon, u) = \mathcal{L}\Bigg(\big((A+\varepsilon b)^n - A^n\big)^q - \sum_{j=0}^{q} w_j \left(A + \frac{j}{q}\varepsilon b \right)^N \Bigg). \tag{2.3}$$

The first term in (2.3) has no powers of ε below q, and its coefficient of ε^q is

$$n^q b^q A^{N-q}.$$

For the second term, expansion in ε gives

$$\sum_{r=0}^{N} \binom{N}{r} A^{N-r} b^r \frac{\varepsilon^r}{q^r} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} j^r.$$

The inner sum is the qth forward difference at zero of the monomial t^r, where j^0 is interpreted as 1, including at $j = 0$. Successive differences lower its degree, and hence

$$\sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} j^r = \begin{cases} 0, & 0 \leq r < q, \\ q!, & r = q. \end{cases}$$

Thus the second term also has no powers below q, while its coefficient of ε^q is

$$\binom{N}{q} \frac{q!}{q^q} b^q A^{N-q} = \frac{(N)_q}{q^q} b^q A^{N-q}.$$

Consequently,

$$H_{n,q} f(u+\varepsilon, u) = \varepsilon^q \left(n^q - \frac{(N)_q}{q^q} \right) \mathcal{L}(b^q A^{N-q}) + O(\varepsilon^{q+1}). \tag{2.4}$$

On the other hand, differentiating the finite polynomial represented by (2.1) yields

$$(B_N f)^{(q)}(u) = (N)_q \mathcal{L}(b^q A^{N-q}).$$

Substitution in (2.4) proves (2.2). Finally, since $N = qn \geq q \geq 2$,

$$0 < (N)_q < N^q = q^q n^q,$$

which proves the asserted positivity. □

Corollary 2.2 (Positivity of the Bernstein derivatives) *Let $q \geq 2$ be an integer and let $f \in C[0,1]$. If (1.5) holds, then, for every $n \geq 1$ and $u \in [0,1]$,*

$$(B_{qn}f)^{(q)}(u) \geq 0.$$

Proof Fix $n \geq 1$ and $u \in (0,1)$. For $0 < \varepsilon < 1-u$, (1.5) gives

$$\frac{H_{n,q}f(u+\varepsilon,u)}{\varepsilon^q} \geq 0.$$

Letting $\varepsilon \downarrow 0$ in (2.2), and using the strict positivity of its coefficient, gives $(B_{qn}f)^{(q)}(u) \geq 0$. This derivative is a polynomial, so the same inequality holds at the endpoints by continuity. □

Proof of Theorem 1.1 Assume first that (1.5) holds. Put $g_n = B_{qn}f$. By Corollary 2.2, $g_n^{(q)}$ is nonnegative on $[0,1]$. If $v \geq 0$, $h > 0$, and $v + qh \leq 1$, repeated use of the fundamental theorem of calculus gives

$$\Delta_h^q g_n(v) = \int_0^h \cdots \int_0^h g_n^{(q)}(v + t_1 + \cdots + t_q)\, dt_1 \cdots dt_q \geq 0. \tag{2.5}$$

Classical Bernstein approximation gives $g_n \to f$ uniformly on $[0,1]$ as $n \to \infty$ [7]. For each such v and h,

$$\left|\Delta_h^q g_n(v) - \Delta_h^q f(v)\right| \leq 2^q \|g_n - f\|_\infty \longrightarrow 0.$$

Passing to the limit in (2.5) shows that $\Delta_h^q f(v) \geq 0$. Thus f is q-monotone by Definition 1.2.

This completes the proof of the converse. For $q \geq 3$, the forward implication is the direct inequality of Abel, Leviatan, and Raşa [2]; for $q = 2$, it is the sharpened convexity inequality proved in [4, 6]. □

3 Consequences and Remarks

The proof above uses only comparison gaps arbitrarily close to the diagonal, and it needs them only along indices tending to infinity. We record the precise form of these observations.

Corollary 3.1 (A common local subsequence) *Let $q \geq 2$ be an integer and let $f \in C[0,1]$. Then f is q-monotone on $[0,1]$ if and only if there is an unbounded set $\mathcal{N} \subseteq \mathbb{N}$ with the following property: for every $n \in \mathcal{N}$ and every $u \in (0,1)$, there is a sequence $\varepsilon_\ell \downarrow 0$ such that*

$$0 < \varepsilon_\ell < 1-u \quad \text{and} \quad H_{n,q}f(u+\varepsilon_\ell, u) \geq 0 \qquad (\ell \geq 1).$$

Proof The forward implication follows from the known direct inequality, with $\mathcal{N} = \mathbb{N}$. Conversely, fix $n \in \mathcal{N}$ and $u \in (0,1)$. Dividing by ε_ℓ^q and applying Lemma 2.1 along the given sequence yields $(B_{qn}f)^{(q)}(u) \geq 0$. As in the proof of Theorem 1.1, it follows that $B_{qn}f$ is q-monotone. Since $\mathcal{N}$ is unbounded, these polynomials converge uniformly to f as $n \to \infty$ through $\mathcal{N}$, and passing each fixed qth forward difference to the limit proves the result. □

Thus it is enough, in particular, that for each $n \in \mathcal{N}$ and $u \in (0,1)$ the inequality hold for all sufficiently small positive ε. Corollary 3.1 shows that even a sequence of comparisons tending to (u,u) suffices. The unbounded set $\mathcal{N}$ is one fixed set, independent of u; no conclusion is asserted here when the tested indices are chosen separately for each comparison pair.

For $q = 2$, Theorem 1.1 takes the following familiar form.

Corollary 3.2 (Characterization of convexity) *Let $f \in C[0,1]$. Then f is convex on $[0,1]$ if and only if*

$$(B_{2n}f)\Big(\frac{x+y}{2}\Big) \geq \sum_{a=0}^{n}\sum_{b=0}^{n} p_{n,a}(x)p_{n,b}(y)f\Big(\frac{a+b}{2n}\Big)$$

for every integer $n \geq 1$ and every $x, y \in [0,1]$.

Proof For $q = 2$, Definition 1.3 gives

$$H_{n,2}f(x,y) = 2\left[(B_{2n}f)\Big(\frac{x+y}{2}\Big) - (T_{n,2,1}f)(x,y)\right].$$

The expression in brackets is symmetric in x and y and vanishes on the diagonal, while continuous 2-monotonicity is equivalent to convexity. The result is therefore the case $q = 2$ of Theorem 1.1. □

References

[1] Abel, U., Leviatan, D.: An extension of Raşa's conjecture to q-monotone functions. Results Math. **75**, Article 180 (2020). https://doi.org/10.1007/s00025-020-01308-y

[2] Abel, U., Leviatan, D., Raşa, I.: Relations between the Bernstein polynomials and q-monotone functions. Results Math. **77**, Article 239 (2022). https://doi.org/10.1007/s00025-022-01764-8

[3] Abel, U., Leviatan, D., Raşa, I.: Remarks on q-monotone functions and the Bernstein polynomials. Results Math. **78**, Article 41 (2023). https://doi.org/10.1007/s00025-022-01810-5

[4] Abel, U., Raşa, I.: A sharpening of a problem on Bernstein polynomials and convex functions. Math. Inequal. Appl. **21**(3), 773–777 (2018). https://doi.org/10.7153/mia-2018-21-55

[5] Graham, R.L., Knuth, D.E., Patashnik, O.: Concrete Mathematics: A Foundation for Computer Science, 2nd edn. Addison-Wesley, Reading, MA (1994)

[6] Komisarski, A., Rajba, T.: A sharpening of a problem on Bernstein polynomials and convex functions and related results. Math. Inequal. Appl. **21**(4), 1125–1133 (2018). https://doi.org/10.7153/mia-2018-21-77

[7] Lorentz, G.G.: Bernstein Polynomials. Mathematical Expositions, vol. 8. University of Toronto Press, Toronto (1953)